

The Decay of Dirac Hair around a Dilaton Black Hole

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The intermediate and late-time behaviour of a massive Dirac field in the background of static spherically symmetric dilaton black hole solutions is investigated. The intermediate asymptotic behaviour of a massive Dirac field depends on the mass parameter as well as the wave number of the mode, while the late-time behaviour has a power law decay rate independent of both.

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I. INTRODUCTION

The late-time behaviour of various fields in the spacetime of a collapsing body is of a great importance in black hole physics. Regardless of details of the collapse or the structure and properties of the collapsing body the resultant black hole can be described by just a few parameters such as mass, charge and angular momentum, *black holes have no hair*. The manner and rate with which the hair of the black hole decays is thus an important question. In what follows we begin by reviewing some of the old and new work on this problem.

Price in Ref.[1] for the first time studied the neutral external perturbations. He found that the late-time behaviour is dominated by the factor $t^{-(2l+3)}$, for each multipole moment l . The decay along null infinity and along the future event horizon the fall off was found [2] to be like that $u^{-(l+2)}$ and $v^{-(l+3)}$, where u and v were the outgoing Eddington-Finkelstein and ingoing Eddington-Finkelstein coordinates. In Ref.[3] the scalar perturbations on the Reissner-Nordström background for the case when $|Q| < M$ was studied. A late time dependence like $t^{-(2l+2)}$ was found, while for $|Q| = M$ the late-time behaviour at fixed r is governed by $t^{-(l+2)}$. Charged scalar hair decayed slower than a neutral one [4]-[6], while the late-time tails in gravitational collapse of fields in the background of Schwarzschild solution was reported by Burko [7]. The intermediate and late-time pattern of hair decay was also considered in Reissner-Nordström background in Ref.[8]. The very late-time tails of massive scalar fields in the Schwarzschild and nearly extremal Reissner-Nordström black holes were the subject of [9], [10]. It was found that the oscillatory tail of a scalar field decays like $t^{-5/6}$ at late times. Power-law tails in the evolution of a charged massless scalar field around the fixed background of a dilaton black hole were studied in [11], while the case of a massive scalar field was treated numerically in Ref.[12]. The analytical proof of the decay pattern both for intermediate and late-time behaviour was presented in [13]. In Ref.[14] the late-time tails of massive scalar field were studied in the spacetime of stationary axisymmetric black hole and it was found that the power law index of $-5/6$ depended neither on multiple mode l nor on the spin rate of the considered black hole. All the above cases involved bosonic fields.

On the fermionic side, the problem of the late-time behaviour of massive Dirac fields were studied in the spacetime of Schwarzschild black hole [15], while in the spacetime of an Reissner-Nordström black hole was analyzed in [16]. The case of the intermediate and the asymptotic behaviour of charged massive Dirac fields in the background of Kerr-Newman black hole was elaborated in Ref.[17].

Interest in unification schemes such as superstring/M-theory has triggered an interest in the decay of the hair of n -dimensional black holes. As far as the n -dimensional static black holes is concerned, the *no-hair* theorem for them is quite well established [18]. The decay mechanism for massless scalar hair around higher n -dimensional Schwarzschild spacetime was worked out in [19]. The late-time tails of massive scalar fields in the spacetime of an n -dimensional static charged black hole was treated in [20] and it was revealed that the intermediate asymptotic behaviour of the considered field had the form $t^{-(l+n/2-1/2)}$. The above pattern of decay was confirmed numerically for the case of $n = 5$ and $n = 6$. One should also mention the results of Ref.[21], where the authors obtained fermion quasi normal modes for massless Dirac fermion in the background of higher dimensional Schwarzschild black hole. Recently, there has been also some efforts to study the late-time behaviour of massive scalar fields in the background of black holes on

brane [22] having in mind the idea that our universe is only a submanifold on which the standard model is confined to, inside a higher dimensional spacetime [23]. On the other hand, the massless fermion excitations on a tensional 3-brane embedded in six-dimensional spacetime were studied in [24].

The main purpose of the present paper is to extend our knowledge of the behaviour of fermionic fields and to clarify what kind of mass-induced behaviours play the dominant role in the asymptotic late-time tails as a result of decaying the massive Dirac hair in the background of a four-dimensional dilaton black hole. That is in a spherically symmetric solution of the low-energy string theory with arbitrary coupling constant α .

The paper is organized as follows. In Sec.II we gave some general remarks concerning behaviour of Dirac spinors in a curved background. Sec.III will be devoted to the analytical studies of intermediate and late-time pattern of decay of the hair in question, while in Sec.IV we conclude our investigations.

II. THE DIRAC EQUATION IN A CURVED SPACETIME

Because of the full spectrum of neutrinos, and their masses and mixing properties is not known, it seems worth while summarizing the general situation. Especially as there has recently been some controversy [26, 27, 28, 29] about the so-called Majorana and Dirac masses and their consequences in gravitational field.

In four spacetime dimensions we can always use a representation in which the gamma matrices are real and we can take the components of all classical fermion fields to take values in a Grassmann algebra over the reals.

The most general Lagrangian for k four-component Majorana fermion ψ^i , $i = 1, 2, \dots, k$ is thus

$$\bar{\psi}^i \not{D} T_{ij} \psi^j - \bar{\psi}^i M_{ij} \psi^j, \quad (1)$$

where $\not{D} = \gamma^\mu \nabla_\mu$ and ∇_μ is the covariant derivative $\nabla_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b$, μ and a are tangent and spacetime indices. There are related by e_μ^a , a basis of orthonormal one-forms. The quantity $\omega_\mu^{ab} \equiv \omega^{ab}$ are the associated connection one-forms satisfying $de^a + \omega_b^a \wedge e^b = 0$. On the other hand, γ^μ are Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. We are using a mainly plus metric signature convention. In four spacetime dimensions, for example, the gamma matrices may be taken to be real. The $4k \times 4k$ matrices $T = \tau_{ij} + \gamma_5 \sigma_{ij}$, $M_{ij} = \mu_{ij} + \gamma_5 \nu_{ij}$ with $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$, $\gamma_5^2 = -1$ and $\tau_{ij}, \mu_{ij}, \nu_{ij}$ symmetric and σ_{ij} anti-symmetric matrices which we take to be independent of time and position.

There is an action on the spinors by $GL(k, \mathbb{C})$ which satisfies

$$\psi^i \rightarrow S^i_j \psi^j, \quad (2)$$

where $S^i_j = \exp(\alpha^i_j + \gamma_5 \beta^i_j)$ and we are thinking of γ_5 as a complex structure on $\mathbb{R}^{4k} \equiv \mathbb{C}^{2k}$. Elements of \mathbb{C}^{2k} are Weyl (or chiral) spinors for which $\gamma_5 = i$. Under this action the result yields

$$T \rightarrow S^\dagger T S, \quad M \rightarrow S^t M S. \quad (3)$$

Using the freedom (3) one may set $T = \text{diag } \pm 1, \pm 1, \dots, \pm 1$. So as to have positive energy we demand that all signs are positive. Now $S \in U(k, \mathbb{C})$ and we may choose it to make M diagonal with real non-negative entries [30]. Thus we arrive at k uncoupled Dirac equations of the form

$$\left(\gamma^\mu \nabla_\mu - m \right) \psi = 0. \quad (4)$$

If one iterates the Dirac equation and uses the cyclic Bianchi identity in a curved space one gets the following:

$$-\nabla^2 \psi + \frac{1}{4} R \psi + m^2 \psi = 0. \quad (5)$$

There is no *gyro-magnetic* coupling between the spin and the Ricci or Riemann tensors [32]. We see in these calculations no sign of the effect claimed in Refs.[26, 27, 29]. This is consistent with the *equivalence principle*, according to which all particles should fall in the same way in a gravitational field. Of course, if the matrices T and M were depended upon position then, things could be different. In the presence of a dilaton and axion field, this might happen. In this paper we shall just consider mass terms. Although the detailed calculations above assume that spacetime is four-dimensional, they are readily extended to higher dimensional spacetimes.

III. THE DECAY OF DIRAC HAIR IN THE BACKGROUND OF A BLACK HOLE SOLUTION

The treatment of fermions in spherically symmetric backgrounds may be greatly simplified by recalling a few basic properties of the Dirac equation. These allow a rapid reduction of the problem to the behaviour of a suitable second order radial equation. We shall begin by giving a discussion valid for all spacetime dimensions n (when $SO(3)$ is replaced by $SO(n-1)$) but our detailed decay results will apply only to the case $n = 4$.

A. Some useful properties of the Dirac operator

As we saw above, we may assume that the massive Dirac equation in a background metric is given by Eq.(4). The basic properties of the Dirac operator $\mathcal{D} = \gamma^\mu \nabla_\mu$ on an n -dimensional manifold that we shall need are

- for a metric product

$$g_{\mu\nu} dx^\mu dx^\nu = g_{ab}(x) dx^a dx^b + g_{mn}(y) dy^m dy^n, \quad (6)$$

it decomposes as a direct sum

$$\mathcal{D} = \mathcal{D}_x + \mathcal{D}_y. \quad (7)$$

- Under a Weyl conformal rescaling given by

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad (8)$$

it follows directly that we have

$$\mathcal{D}\psi = \Omega^{-\frac{1}{2}(n+1)} \tilde{\mathcal{D}}\tilde{\psi}, \quad \psi = \Omega^{-\frac{1}{2}(n-1)} \tilde{\psi}. \quad (9)$$

For a conformo-static metric of the form

$$ds^2 = -A^2 dt^2 + \Phi^2 dx^i dx^i, \quad (10)$$

where $A = A(x^i)$ and $\Phi = \Phi(x^i)$, $i = 1, 2 \dots n-1$, we write

$$ds^2 = A^2 \left(-dt^2 + \left(\frac{\Phi}{A} \right)^2 dx^i dx^i \right), \quad (11)$$

and consequently find the following:

$$\mathcal{D}\psi = A^{-\frac{1}{2}(n+2)} \left(\gamma^0 \partial_t + \tilde{\mathcal{D}} \right) \tilde{\psi}, \quad (12)$$

where $\tilde{\mathcal{D}}$ is the Dirac operator of the metric $\left(\frac{\Phi}{A} \right)^2 dx^i dx^j$ and $\tilde{\psi} = A^{\frac{1}{2}(n-1)} \psi$. Now we use the conformal property again. One obtains the relation

$$\tilde{\mathcal{D}}\tilde{\psi} = \left(\frac{A}{\Phi} \right)^{\frac{1}{2}(n-1)} \gamma^i \partial_i \tilde{\psi}, \quad (13)$$

with $\tilde{\psi} = \left(\frac{A}{\Phi} \right)^{\frac{1}{2}(n-2)} \tilde{\tilde{\psi}}$.

Since every spherically symmetric metric is conformally flat, a special case of the theory above, is a static metric of the form given by

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\Sigma_{n-2}^2, \quad (14)$$

where $A = A(r)$, $B = B(r)$, $C = C(r)$ are functions only of the radial variable r , and the *transverse* metric $d\Sigma_{n-2}^2$ depends neither on t nor on r .

Let us suppose now, that Ψ is a spinor eigenfunction on the $(n-2)$ -dimensional *transverse* manifold Σ . Namely, Ψ satisfies the relation of the form as

$$\mathcal{D}_\Sigma \Psi = \lambda \Psi. \quad (15)$$

By virtue of the properties given above one may assume that the following is satisfied:

$$\mathcal{D}\psi = m\psi, \quad (16)$$

and set what follows:

$$\psi = \frac{1}{A^{\frac{1}{2}}} \frac{1}{C^{\frac{(n-2)}{2}}} \chi \otimes \Psi. \quad (17)$$

It can be verified by the direct calculations that Eq.(16) provides the result as

$$(\gamma^0 \partial_t + \gamma^1 \partial_y) \chi = A(m - \frac{\lambda}{C}) \chi. \quad (18)$$

where we have denoted

$$dy = \frac{B}{A} dr, \quad (19)$$

the *radial optical distance* (i.e., the Regge-Wheeler radial coordinate). Gamma matrices γ^0, γ^1 satisfy the Clifford algebra in two spacetime dimensions.

An identical result may be obtained if a Yang-Mills gauge field A_μ is present on the transverse manifold Σ , but now one gets

$$\mathcal{D}_{\Sigma, A_\mu} \Psi = \lambda \Psi, \quad (20)$$

where $\mathcal{D}_{\Sigma, A_\mu}$ is the Dirac operator twisted by the connection A_μ .

Assuming that $\psi \propto e^{-i\omega t}$, one obtains the second order equation for χ

$$\frac{d^2 \chi}{dy^2} + \omega^2 \chi = A^2(m - \frac{\lambda}{C})^2 \chi. \quad (21)$$

In what follows, the detailed form of the spinor harmonics and the eigenvalues will not be important.

B. The Background

In four spacetime dimensions, the action for the dilaton gravity with arbitrary coupling constant implies

$$S = \int d^4 x \sqrt{-g} \left[R - 2\nabla^\mu \phi \nabla_\mu \phi - e^{-2\alpha\phi} F_{\mu\nu} F^{\mu\nu} \right], \quad (22)$$

where ϕ is the dilaton field, α coupling constant while $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ is the strength of $U(1)$ gauge field. The static spherically symmetric solution of the equations of motion are given by the following line element:

$$ds^2 = - \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} dt^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}} + R^2(r) d\Omega^2, \quad (23)$$

where $R^2(r) = r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}$, while r_+ and r_- are related to the mass M and the electric charge Q of the black hole

$$e^{-2\alpha\phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad 2M = r_+ + \frac{1-\alpha^2}{1+\alpha^2} r_-, \quad Q^2 = \frac{r_- r_+}{1+\alpha^2}. \quad (24)$$

The metric is asymptotically flat in the sense that the spacetime contains an initial data set $(\Sigma_{end}, g_{ij}, K_{ij})$ with gauge fields such that Σ_{end} is diffeomorphic to \mathbf{R}^3 minus a ball and the following asymptotic conditions are fulfilled:

$$|g_{ij} - \delta_{ij}| + r|\partial_a g_{ij}| + \dots + r^k |\partial_{a_1 \dots a_k} g_{ij}| + r|K_{ij}| + \dots + r^k |\partial_{a_1 \dots a_k} K_{ij}| \leq \mathcal{O}\left(\frac{1}{r}\right), \quad (25)$$

$$|F_{\alpha\beta}| + r|\partial_a F_{\alpha\beta}| + \dots + r^k |\partial_{a_1 \dots a_k} F_{\alpha\beta}| \leq \mathcal{O}\left(\frac{1}{r^2}\right), \quad (26)$$

$$\phi = \phi_0 + \mathcal{O}\left(\frac{1}{r}\right), \quad (27)$$

where K_{ij} is the exterior curvature, ϕ_0 is a constant value of the scalar field.

C. Spinor No-Hair theorems

The properties of static spinor fields around Schwarzschild and Kerr black holes and the consequent *no-hair* properties have been investigated by many people including [33, 34, 35, 36]. The basic idea is to study solutions of the static Dirac on the background of the black hole. One either considers the case when there are no fermionic sources outside the horizon or one constructs a Green function. In the massless static spherical case it is clear from our work above that this is equivalent to solving the flat space Dirac equation [37]

$$\gamma^i \partial_i \tilde{\psi} = 0. \quad (28)$$

This may have regular solutions on the horizon. On the other hand, in our case, we have the following:

$$\psi = \frac{1}{A\Phi^{\frac{1}{2}(n-2)}} \tilde{\psi}. \quad (29)$$

On the horizon $A = 0$ and, unless the solution is extreme, $\Phi \neq 0$ [37]. The extreme case is exceptional because $\frac{1}{\Phi} = 0$ at the horizon in such a way that the spinor ψ remains finite [37]. Of course, in the non-extreme case, one should check that some scalar spinorial invariant blows up, but this can easily be done.

D. Decay of Fermionic Hair

We shall now analyze the time evolution of massive Dirac spinor field in the background of dilaton black hole by means of the spectral decomposition method. In [8],[25] it was shown that the asymptotic tail is connected with the existence of a branch cut situated along the interval $-m \leq \omega \leq m$. An oscillatory inverse power-law behaviour of massive Dirac field arises from the integral of Green function $\tilde{G}(y, y'; \omega)$ around branch cut. The time evolution of massive Dirac field may be written in the following form:

$$\chi(y, t) = \int dy' \left[G(y, y'; t) \chi_t(y', 0) + G_t(y, y'; t) \chi(y', 0) \right], \quad (30)$$

for $t > 0$, where the Green's function $G(y, y'; t)$ is given by the relation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + V \right] G(y, y'; t) = \delta(t) \delta(y - y'). \quad (31)$$

In what follows, our main task will be to find the dilaton black hole Green function. Using the Fourier transform [25] $\tilde{G}(y, y'; \omega) = \int_0^\infty dt G(y, y'; t) e^{i\omega t}$ one can reduce equation (31) to an ordinary differential equation. The Fourier's transform is well defined for $\text{Im } \omega \geq 0$, while the corresponding inverse transform yields

$$G(y, y'; t) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\omega \tilde{G}(y, y'; \omega) e^{-i\omega t}, \quad (32)$$

for some positive number ϵ . By virtue of the above the Fourier's component of the Green's function $\tilde{G}(y, y'; \omega)$ can be written in terms of two linearly independent solutions for homogeneous equation. Namely, one has

$$\left(\frac{d^2}{dy^2} + \omega^2 - \tilde{V} \right) \chi_i = 0, \quad i = 1, 2, \quad (33)$$

where $\tilde{V} = A^2 \left(m - \frac{\lambda}{C} \right)^2$.

The boundary conditions for χ_i are described by purely ingoing waves crossing the outer horizon H_+ of the static charged black hole $\chi_1 \simeq e^{-i\omega y}$ as $y \rightarrow -\infty$. On the other hand, χ_2 should be damped exponentially at i_+ , namely $\chi_2 \simeq e^{-\sqrt{m^2 - \omega^2} y}$ at $y \rightarrow \infty$.

Let us assume that the observer and the initial data are situated far away from the considered black hole. In order to rewrite Eq.(33) in a more convenient form we change variables

$$\chi_i = \frac{\xi}{\left(1 - \frac{r_+}{r} \right)^{1/2} \left(1 - \frac{r_-}{r} \right)^{\frac{1-\alpha^2}{2(1+\alpha^2)}}}, \quad (34)$$

where $i = 1, 2$. Then, one can expand Eq.(33) as a power series of r_{\pm}/r neglecting terms of order $\mathcal{O}((\omega/r)^2)$ and higher. Under this assumption we reach to the following:

$$\frac{d^2}{dr^2}\xi + \left[\omega^2 - m^2 + \frac{2\omega^2(r_+ + \alpha_1 r_-) - m^2(r_+ + \alpha_1 r_-) + 2\lambda m(1 + r_+)}{r} - \frac{\lambda^2 - 2\lambda m r_-(\alpha_1 + \alpha_2) + m^2 \alpha_1 r_+ r_-}{r^2} \right] \xi = 0, \quad (35)$$

where we have denoted $\alpha_1 = \frac{1-\alpha^2}{1+\alpha^2}$ and $\alpha_2 = \frac{2\alpha^2}{1+\alpha^2}$.

It can be verified that Eq.(35) may be solved in terms of Whittaker's functions. Consequently, two basic solutions are needed to construct the Green function, with the condition that $|\omega| \geq m$. Namely, the Whittaker's functions $\tilde{\chi}_1 = M_{\delta, \tilde{\mu}}(2\tilde{\omega}r)$ and $\tilde{\chi}_2 = W_{\delta, \tilde{\mu}}(2\tilde{\omega}r)$ have the following parameters:

$$\begin{aligned} \tilde{\mu} &= \sqrt{1/4 + \lambda^2 - 2\lambda m r_- + m^2 \alpha_1 r_+ r_-}, \\ \delta &= \frac{\omega^2(r_+ + \alpha_1 r_-) + \lambda m(1 + r_+) - \frac{m^2}{2}(r_+ + \alpha_1 r_-)}{\tilde{\omega}}, \\ \tilde{\omega}^2 &= m^2 - \omega^2. \end{aligned} \quad (36)$$

On the other hand, the spectral Green function takes the form as

$$\begin{aligned} G_c(r, r'; t) &= \frac{1}{2\pi} \int_{-m}^m dw \left[\frac{\tilde{\chi}_1(r, \tilde{\omega} e^{\pi i}) \tilde{\chi}_2(r', \tilde{\omega} e^{\pi i})}{W(\tilde{\omega} e^{\pi i})} - \frac{\tilde{\chi}_1(r, \tilde{\omega}) \tilde{\chi}_2(r', \tilde{\omega})}{W(\tilde{\omega})} \right] e^{-i\omega t} \\ &= \frac{1}{2\pi} \int_{-m}^m d\omega f(\tilde{\omega}) e^{-i\omega t}, \end{aligned} \quad (37)$$

where $W(\tilde{\omega})$ is the Wronskian.

Further, we focus our attention on the intermediate asymptotic decay of the massive Dirac hair, i.e., in the range of parameters $M \ll r \ll t \ll M/(mM)^2$. The intermediate asymptotic contribution to the Green function integral gives the frequency equal to $\tilde{\omega} = \mathcal{O}(\sqrt{m/t})$, which in turns implies that $\delta \ll 1$. Having in mind that δ results from the $1/r$ term in the massive scalar field equation of motion, it depicts the effect of backscattering off the spacetime curvature and in the case under consideration the backscattering is negligible. Taking into account all the above and the fact that $\tilde{\omega}r \ll 1$ and $M(a, b, z) = 1$ as z tends to zero, we obtain the resulting expression for spectral Green function

$$G_c(r, r'; t) = \frac{2^{3\tilde{\mu}-\frac{3}{2}} \Gamma(-2\tilde{\mu}) \Gamma(\frac{1}{2} + \tilde{\mu}) \Gamma(\tilde{\mu} + 1)}{\tilde{\mu} \sqrt{\pi} \tilde{\mu} \Gamma(2\tilde{\mu}) \Gamma(\frac{1}{2} - \tilde{\mu})} \left(1 + e^{(2\tilde{\mu}+1)\pi i} \right) (rr')^{\frac{1}{2}+\tilde{\mu}} \left(\frac{m}{t} \right)^{\frac{1}{2}+\tilde{\mu}} J_{\frac{1}{2}+\tilde{\mu}}(mt). \quad (38)$$

In the limit when $t \gg 1/m$ it implies

$$G_c(r, r'; t) = \frac{2^{3\tilde{\mu}-1} \Gamma(-2\tilde{\mu}) \Gamma(\frac{1}{2} + \tilde{\mu}) \Gamma(\tilde{\mu} + 1)}{\tilde{\mu} \sqrt{\pi} \tilde{\mu} \Gamma(2\tilde{\mu}) \Gamma(\frac{1}{2} - \tilde{\mu})} \left(1 + e^{(2\tilde{\mu}+1)\pi i} \right) (rr')^{\frac{1}{2}+\tilde{\mu}} m^{\tilde{\mu}} t^{-1-\tilde{\mu}} \cos(mt - \frac{\pi}{2}(\tilde{\mu} + 1)). \quad (39)$$

Eq.(39) depicts the oscillatory inverse power-law behaviour. We remark that in our case the intermediate times of the power-law tail depends only on $\tilde{\mu}$ which in turn is a function of the multiple number of the wave modes.

The different pattern of decay is expected when $\kappa \gg 1$, for the late-time behaviour, when the backscattering off the curvature is important. Consequently, $f(\tilde{\omega})$ when $\kappa \gg 1$ may be rewritten in the following form:

$$\begin{aligned} f(\tilde{\omega}) &= \frac{\Gamma(1 + 2\tilde{\mu}) \Gamma(1 - 2\tilde{\mu})}{2\tilde{\mu}} (rr')^{\frac{1}{2}} \left[J_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) J_{-2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) - I_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) I_{-2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) \right] \\ &+ \frac{(\Gamma(1 + 2\tilde{\mu}))^2 \Gamma(-2\tilde{\mu}) \Gamma(\frac{1}{2} + \tilde{\mu} - \delta)}{2\tilde{\mu} \Gamma(2\tilde{\mu}) \Gamma(\frac{1}{2} - \tilde{\mu} - \delta)} (rr')^{\frac{1}{2}} \delta^{-2\tilde{\mu}} \left[J_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) J_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) \right. \\ &\left. + e^{(2\tilde{\mu}+1)\pi i} I_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) I_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) \right], \end{aligned} \quad (40)$$

where we have used the limit $M_{\delta, \tilde{\mu}}(2\tilde{\omega}r) \approx \Gamma(1 + 2\tilde{\mu})(2\tilde{\omega}r)^{\frac{1}{2}} \delta^{-\tilde{\mu}} J_{\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r})$. The first part of the above Eq.(40) the late time tail is proportional to t^{-1} and it occurs that we shall concentrate on the second term of the right-hand side of Eq.(40). It turned out that for the case when $\kappa \gg 1$ it may be rewritten in the form as

$$G_{c(2)}(r, r'; t) = \frac{M}{2\pi} \int_{-m}^m dw e^{i(2\pi\delta - \omega t)} e^{i\varphi}, \quad (41)$$

where we have used the following definition:

$$e^{i\varphi} = \frac{1 + (-1)^{2\tilde{\mu}} e^{-2\pi i\delta}}{1 + (-1)^{2\tilde{\mu}} e^{2\pi i\delta}}. \quad (42)$$

On the other hand, M yields

$$M = \frac{(\Gamma(1 + 2\tilde{\mu}))^2 \Gamma(-2\tilde{\mu})}{2\tilde{\mu} \Gamma(2\tilde{\mu})} (rr')^{\frac{1}{2}} \left[J_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) J_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) + I_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r}) I_{2\tilde{\mu}}(\sqrt{8\delta\tilde{\omega}r'}) \right]. \quad (43)$$

At very late time both terms $e^{i\omega t}$ and $e^{2\pi\delta}$ are rapidly oscillating. From this fact it follows directly that the spinor waves are mixed states consisting of the states with multipole phases backscattered by spacetime curvature, which most of them cancel with each others which have the inverse phase. Thus, one can find the value of G_c (2) by means of the saddle point method. The saddle point integration allows us to evaluate the accurate value of the asymptotic behaviour. Namely, it could be found that the value $2\pi\delta - \omega t$ is stationary at the value of ω equal to the following:

$$a_0 = \left[\frac{\pi (\omega^2 (r_+ + \alpha_1 r_-) + \lambda m(1 + \alpha_1 r_-) - \frac{m^2}{2}(r_+ + \alpha_1 r_-))}{\sqrt{2}m} \right]^{\frac{1}{3}}, \quad (44)$$

Evaluating Eq.(41) by means of the saddle point integration we achieve finally to the form of the spectral Green function for massive Dirac spinor hair. It implies

$$G_c(r, r'; t) = \frac{2\sqrt{2}}{\sqrt{3}} m^{2/3} (\pi)^{\frac{5}{6}} \left[2m^2(r_+ + \alpha_1 r_-) + 2\lambda m(1 + r_+) - m^2(r_+ + \alpha_1 r_-) \right]^{\frac{1}{3}} (mt)^{-\frac{5}{6}} \sin(mt) \tilde{\chi}(r, m) \tilde{\chi}(r', m). \quad (45)$$

The above equation provides the main result of our calculations. It illustrates the fact that the late-time asymptotic decay pattern of massive Dirac hair in the background of spherically symmetric dilaton black hole is proportional to $-5/6$.

IV. CONCLUSIONS

In this paper we have treated the problem of the asymptotic tail behaviour of a free Dirac field in the spacetime of a spherically symmetric charged black hole solution of dilaton gravity with arbitrary coupling constant α . This theory is related to the low-energy limit of the heterotic string theory and on its own is a generalization of electromagnetism by adding scalar field ϕ dilaton and coupling constant between $U(1)$ gauge field and scalar field.

The resultant intermediate asymptotic behaviour depends on the field parameter mass as well as the wave number of the mode. But this is not the final pattern of decay of the massive Dirac hair. Resonance backscattering off the spacetime curvature dominates at late times. We have calculated analytically that the pattern of decay in question is proportional to $t^{-\frac{5}{6}}$. The same result one gets studying the late-time behaviour of free massive scalar fields in the same background. One should remark that the above considerations are also applicable to the case of extremal dilaton black hole, i.e., to the case when $r_+ = r_-$. Thus having in mind Eq.(45) one gets the exact form of the spectral Green function for the late-time behaviour of massive Dirac hair for the extremal dilaton black hole in the theory with arbitrary coupling constant α .

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